

# Waves in fractional Zener type viscoelastic media

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## Abstract

Classical wave equation is generalized for the case of viscoelastic materials obeying fractional Zener model instead of Hooke's law. Cauchy problem for such an equation is studied: existence and uniqueness of the fundamental solution is proven and solution is calculated.

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## 1 Introduction

The need for a use of derivatives of arbitrary real (or even complex) order, also called fractional derivatives, is permanently underlined by various applied problems. Although, the idea about differentiating with respect to noninteger order originates from the work of Leibnitz and Euler at the turn of the 18th century, the last 30 years have brought a real expansion of fractional calculus. Certainly, it is due to great number of possibilities for its application in many diverse fields such as mechanics, viscoelasticity, automatic control, signal processing, stochastic and finance, biomedical engineering, etc. For a detailed exposition of the theory of fractional calculus we refer to [16] or [19].

In this paper we study existence and uniqueness of solutions for the system

$$\begin{aligned}\frac{\partial}{\partial x}\sigma(x, t) &= \frac{\partial^2}{\partial t^2}u(x, t), \\ \sigma(x, t) + \tau_0 D_t^\alpha \sigma(x, t) &= \varepsilon(x, t) + {}_0D_t^\alpha \varepsilon(x, t), \\ \varepsilon(x, t) &= \frac{\partial}{\partial x}u(x, t),\end{aligned}\tag{1}$$

where  $x \in \mathbb{R}$ ,  $t > 0$ ,  $\sigma$ ,  $u$  and  $\varepsilon$  are stress, displacement and strain, respectively, considered as functions of  $x$  and  $t$ ,  $0 < \tau < 1$  is a constant and  ${}_0D_t^\alpha$ ,  $0 \leq \alpha < 1$ , denotes the left Riemann-Liouville operator of fractional differentiation of order  $\alpha$ . This system describes waves in viscoelastic media.

System (1) is subjected to initial conditions

$$\begin{aligned}u(x, 0) &= u_0(x), & \frac{\partial}{\partial t}u(x, 0) &= v_0(x), \\ \sigma(x, 0) &= 0, & \varepsilon(x, 0) &= 0,\end{aligned}\tag{2}$$

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as well as boundary conditions

$$\lim_{x \rightarrow \pm\infty} u(x, t) = 0, \quad \lim_{x \rightarrow \pm\infty} \sigma(x, t) = 0. \quad (3)$$

As we shall show, system (1) can be reduced to the equation

$$\frac{\partial^2}{\partial t^2} u(x, t) = L(t) \frac{\partial^2}{\partial x^2} u(x, t), \quad x \in \mathbb{R}, t > 0, \quad (4)$$

which will be called wave equation for fractional Zener type viscoelastic media or shortly FZWE (Fractional Zener Wave Equation). Here  $L$  denotes a linear operator (of convolution type) acting on  $\mathcal{S}'(\mathbb{R})$ , whose explicit form is given by

$$L(t) = \mathcal{L}^{-1} \left( \frac{1 + s^\alpha}{1 + \tau s^\alpha} \right) *_t.$$

In fact, equation (4) will be the subject of our consideration. System (1), and in particular (4), generalize classical wave equation.

Equation that describes waves occurring in the elastic media are obtained by using basic equations of elasticity (see [2]). Our interest is restricted to an infinite elastic rod (one dimensional body), positioned along  $x$  axis, that is not under influence of body forces. Equations of elasticity then read

$$\frac{\partial}{\partial x} \sigma(x, t) = \rho \frac{\partial^2}{\partial t^2} u(x, t) \quad (5)$$

$$\sigma(x, t) = E \varepsilon(x, t) \quad (6)$$

$$\varepsilon(x, t) = \frac{\partial}{\partial x} u(x, t) \quad (7)$$

where  $x \in \mathbb{R}$ ,  $t > 0$ ,  $\sigma$ ,  $u$  and  $\varepsilon$  are stress, displacement and strain, as above,  $\rho = \text{const.}$  is the density of the media and  $E = \text{const.}$  is the Young modulus of elasticity. Equation (5) is the equilibrium equation and it is the consequence of the Second Newton Law. Equation (6) is the constitutive equation and in the case of elastic media it is known as the Hooke law. Equation (7) is the strain measure for the local small deformations, i.e. connection between strain and displacement. Wave equation is now obtained by substituting equation (7) into (6) and subsequently (6) into (5). It reads

$$\frac{\partial^2}{\partial t^2} u(x, t) = c^2 \frac{\partial^2}{\partial x^2} u(x, t), \quad c = \sqrt{\frac{E}{\rho}}. \quad (8)$$

Newly introduced parameter  $c$  can be physically interpreted as the wave speed.

The first authors who introduced the fractional Zener model were M. Caputo and F. Mainardi [6, 7]. Such model was also considered in [17] but without using fractional calculus. A similar problem of stress waves in a viscoelastic medium was investigated in [9] by using the inversion formula of the classical Laplace transform. Wave equation (8) can be generalized in the framework of fractional calculus in several ways. One of the possibilities is to substitute second order partial derivative with respect to time by fractional one, as it has been done in [13] for one dimension, and in [10] and [11] for  $d$  dimensions,  $d \in \{1, 2, 3\}$ . Time partial derivative can also be replaced with either two fractional derivatives of different order, as in [3], or by distributed order fractional derivative (integral over given range of order of fractional derivatives), as in [12] and [14] for  $\alpha \in (0, 1)$ , and in [4] and [5] for  $\alpha \in [0, 2]$ . In each of these papers the intention was to stress similarity between wave and diffusion equation, since by allowing the order of fractional derivative to be  $\alpha \in [0, 2]$  both equations can be obtained as special cases.

In this article wave equation (8) is generalized for the case of viscoelastic media described by fractional Zener model. Generalization is conducted so that constitutive equation (6) is changed, since Hooke's law is inappropriate to describe viscoelastic media. Other two equations, namely equation of motion of deformable body (5) and relation between strain and displacement (7),

are valid for any type of deformable body, with mentioned restrictions, therefore they remain unchanged. Similar generalization was done in [18] in case of bounded domain.

The paper is organized as follows. In Section 2 we recall basic notions and fix notation which will be used throughout this paper. In Section 3 we introduce dimensionless quantities, derive (1) and (4) and set up the Cauchy problem (4), (2) in the setting of distribution theory. Section 4 is devoted to examination of the Cauchy problem (4), (2) where we prove a theorem on existence and uniqueness of its fundamental solution. In Section 5 we give a numerical example, which illustrates the behavior of the fundamental solution for the specific choice of the parameters  $\alpha$  and  $\tau$ , as well as initial conditions.

## 2 Mathematical preliminaries

Let  $\Omega$  be an open set in  $\mathbb{R}^n$ . The space of distributions will be denoted by  $\mathcal{D}'(\Omega)$  and the space of Schwartz tempered distributions by  $\mathcal{S}'(\mathbb{R}^n)$ .  $\mathcal{D}'_+(\mathbb{R})$  and  $\mathcal{S}'_+(\mathbb{R})$  are the subspaces of  $\mathcal{D}'(\mathbb{R})$  and  $\mathcal{S}'(\mathbb{R})$  respectively, containing distributions supported on  $[0, \infty)$ . Since we shall mostly work with functions and distributions depending on two variables,  $u = u(x, t)$ , we introduce  $\mathcal{S}'(\mathbb{R} \times \mathbb{R}_+)$  to be the space of all distributions  $u \in \mathcal{S}'(\mathbb{R}^2)$ , which vanish for  $t < 0$ .

We shall also need the set  $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$ .

Let  $t \in [0, a]$ ,  $a > 0$ , and  $y \in AC([0, a])$ . The left Riemann-Liouville fractional derivative of order  $\alpha \in [0, 1)$ ,  ${}_0D_t^\alpha y$ , is defined as

$${}_0D_t^\alpha y(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{y(\zeta)}{(t-\zeta)^\alpha} d\zeta,$$

where  $\Gamma$  is the Euler gamma function.

In the distributional setting, one introduces a family  $\{f_\alpha\}_{\alpha \in \mathbb{R}} \in \mathcal{D}'_+(\mathbb{R})$  as

$$f_\alpha(t) = \begin{cases} H(t) \frac{t^{\alpha-1}}{\Gamma(\alpha)}, & \alpha > 0, \\ \frac{d^N}{dt^N} f_{\alpha+N}(t), & \alpha \leq 0, \alpha + N > 0, N \in \mathbb{N} \end{cases},$$

where  $H$  is the Heaviside function. Then  $f_\alpha*$  is a convolution operator acting on  $\mathcal{D}'_+(\mathbb{R})$  (also,  $f_\alpha* : \mathcal{S}'_+(\mathbb{R}) \rightarrow \mathcal{S}'_+(\mathbb{R})$ ). For  $\alpha < 0$  it is called the operator of fractional differentiation. Moreover, for  $y \in AC([0, a])$  it coincides with the left Riemann-Liouville fractional derivative, i.e.

$${}_0D_t^\alpha y = f_{-\alpha} * y.$$

For  $y \in \mathcal{S}'(\mathbb{R})$  the Fourier transform is defined as

$$\langle \mathcal{F}y, \varphi \rangle = \langle y, \mathcal{F}\varphi \rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}),$$

where for  $\varphi \in \mathcal{S}(\mathbb{R})$

$$\mathcal{F}\varphi(\xi) = \hat{\varphi}(\xi) = \int_{-\infty}^{\infty} \varphi(x) e^{-i\xi x} dx, \quad \xi \in \mathbb{R}.$$

Let  $y \in \mathcal{D}'_+(\mathbb{R})$  such that  $e^{-\xi t} y \in \mathcal{S}'(\mathbb{R})$ , for all  $\xi > a > 0$ . Then the Laplace transform of  $y$  is defined by

$$\mathcal{L}y(s) = \tilde{y}(s) = \mathcal{F}(e^{-\xi t} y)(\eta), \quad s = \xi + i\eta.$$

It is well known that the function  $\mathcal{L}y$  is holomorphic in the half plane  $\operatorname{Re} s > a$  (see e.g. [8] or [21]). In particular, for  $y \in L^1(\mathbb{R})$  such that  $y(t) = 0$ , for  $t < 0$ , and  $|y(t)| \leq Ae^{at}$  ( $a, A > 0$ ) the Laplace transform is

$$\mathcal{L}y(s) = \int_0^\infty y(t) e^{-st} dt, \quad \operatorname{Re} s > 0.$$

We shall need the following properties:  $y, y_1, y_2 \in \mathcal{S}'(\mathbb{R})$

$$\begin{aligned}\mathcal{F}\left[\frac{d^n}{dx^n}y\right](\xi) &= (i\xi)^n \mathcal{F}y(\xi), \\ \mathcal{L}[{}_0D_t^\alpha y](s) &= s^\alpha \mathcal{L}y(s), \\ \mathcal{L}[y_1 * y_2](s) &= \mathcal{L}y_1(s) \cdot \mathcal{L}y_2(s), \\ \mathcal{L}\delta(s) &= 1.\end{aligned}$$

In order to introduce the inverse Laplace transform we recall from [21] the following: Let  $Y$  be a holomorphic function in the half plane  $\operatorname{Re} s > a$  such that  $|Y(s)| \leq A \frac{(1+|s|^m)}{|\operatorname{Re} s|^k}$ ,  $m, k \in \mathbb{R}$ . Then there exists a distribution  $y \in \mathcal{S}'_+(\mathbb{R})$  such that  $\mathcal{L}y = Y$ , and

$$y(t) = \mathcal{L}^{-1}Y(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} Y(s) e^{st} ds, \quad t > 0.$$

The notion of fundamental solution is introduced as follows (see e.g. [21]). Let  $P$  be a linear partial integro-differential operator with constant coefficients. A fundamental solution of  $P$ , denoted by  $E$ , is a solution to the equation  $Pu = \delta$ . Once the fundamental solution is determined one finds a solution to  $Pu = f$  as  $u = E * f$ .

The Cauchy problem for the second order linear partial integro-differential operator with constant coefficients  $P$  is given by

$$Pu(x, t) = f(x, t) \tag{9}$$

$$u(x, 0) = u_0(x), \quad \frac{\partial}{\partial t} u(x, 0) = u_1(x), \tag{10}$$

where  $f$  is continuous for  $t \geq 0$ ,  $u_0 \in C^1(\mathbb{R})$  and  $u_1 \in C(\mathbb{R})$ . A classical solution  $u(x, t)$  to the Cauchy problem (9-10) is of class  $C^2$  for  $t > 0$ , of class  $C^1$  for  $t \geq 0$ , satisfies equation (9) for  $t > 0$ , and initial conditions (10) when  $t \rightarrow 0$ . If functions  $u$  and  $f$  are continued by zero for  $t < 0$ , then the following equation is satisfied in  $\mathcal{D}'(\mathbb{R}^2)$ :

$$Pu = f(x, t) + u_0(x)\delta'(t) + u_1(x)\delta(t). \tag{11}$$

The classical solutions of the Cauchy problem (9-10) are among those solutions of equation (11) that vanish for  $t < 0$ . Therefore, the problem of finding generalized solutions (in  $\mathcal{D}'(\mathbb{R}^2)$ ) of the equation (11) that vanish for  $t < 0$  will be called *generalized Cauchy problem for the operator  $P$* . If there is a fundamental solution  $E$  of the operator  $P$  and if  $f \in \mathcal{D}'(\mathbb{R}^2)$  vanishes for  $t < 0$  then there exists a unique solution to corresponding generalized Cauchy problem and is given by

$$u = E * (f(x, t) + u_0(x)\delta'(t) + u_1(x)\delta(t)).$$

Again, we refer to [8], [20] or [21] for more details.

### 3 Set up of the Cauchy problem

In order to obtain (1), we introduce dimensionless coordinates in the system (5-7), where (6) is modified by the fractional Zener model of the viscoelastic body:

$$\sigma(x, t) + \tau_{\sigma 0} D_t^\alpha \sigma(x, t) = E[\varepsilon(x, t) + \tau_{\varepsilon 0} D_t^\alpha \varepsilon(x, t)], \quad x \in \mathbb{R}, t > 0, \tag{12}$$

where  $\tau_\sigma, \tau_\varepsilon$  are relaxation times satisfying  $\tau_\varepsilon > \tau_\sigma > 0$ . The latter condition follows from the Second Law of Thermodynamics (see [1]).

We subject to the system (5), (7) and (12) initial conditions

$$\begin{aligned}u(x, 0) &= u_0(x), & \frac{\partial}{\partial t} u(x, 0) &= v_0(x), \\ \sigma(x, 0) &= 0, & \varepsilon(x, 0) &= 0,\end{aligned}$$

where  $x \in \mathbb{R}$ ,  $u_0$  and  $v_0$  are initial displacement and velocity. Note that there are no initial stress and strain. We also supply boundary conditions

$$\lim_{x \rightarrow \pm\infty} u(x, t) = 0, \quad \lim_{x \rightarrow \pm\infty} \sigma(x, t) = 0, \quad t \geq 0.$$

We will need the following lemma.

**Lemma 3.1.** *Let  $y \in AC([0, a])$ ,  $a > 0$ ,  $0 \leq \alpha < 1$  and  $T^* > 0$ . Let  $t$  and  $y$  be transformed as  $(\bar{t}, \bar{y}) = (\frac{t}{T^*}, y)$ , then the left Riemann-Liouville fractional derivative  ${}_0D_t^\alpha y(t)$ ,  $t \geq 0$ , is transformed as follows*

$${}_0D_{\bar{t}}^\alpha \bar{y}(\bar{t}) = (T^*)^\alpha {}_0D_t^\alpha y(t).$$

**Proof.** Since  $\bar{y}(\bar{t}) = y(t) = y(\bar{t}T^*)$ , we have

$$\begin{aligned} {}_0D_{\bar{t}}^\alpha \bar{y}(\bar{t}) &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{d\bar{t}} \int_0^{\bar{t}} \frac{\bar{y}(\xi)}{(\bar{t}-\xi)^\alpha} d\xi \\ &= \frac{1}{\Gamma(1-\alpha)} T^* \frac{d}{dt} \int_0^{\frac{t}{T^*}} \frac{y(\xi T^*)}{(\frac{t}{T^*}-\xi)^\alpha} d\xi \\ &= (T^*)^\alpha \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{y(z)}{(t-z)^\alpha} dz \\ &= (T^*)^\alpha {}_0D_t^\alpha y(t). \end{aligned}$$

□

Let  $0 \leq \alpha < 1$ ,  $\tau_\varepsilon > \tau_\sigma > 0$  and  $E$  be constants appearing in (12), and let  $\rho$  be the constant in (5). Set  $T^* = \sqrt[3]{\tau_\varepsilon}$  and  $X^* = \sqrt[3]{\tau_\varepsilon} \sqrt{\frac{E}{\rho}}$ . Then  $X^*$  and  $T^*$  are constants (measured in meters and seconds respectively) which satisfy  $\left(\frac{X^*}{T^*}\right)^2 \frac{\rho}{E} = 1$  and  $(T^*)^\alpha = \frac{1}{\tau_\varepsilon}$ .

We introduce dimensionless quantities in (5), (12) and (7) as

$$\bar{x} = \frac{x}{X^*}; \quad \bar{t} = \frac{t}{T^*}; \quad \bar{u} = \frac{u}{X^*}; \quad \bar{\sigma} = \frac{\sigma}{E}$$

and

$$\bar{u}_0 = \frac{u_0}{X^*}; \quad \bar{v}_0 = \frac{v_0}{X^*} T^*; \quad \bar{\tau} = \frac{\tau_\sigma}{\tau_\varepsilon}$$

and obtain system (1-3). Note that  $\varepsilon$  is already dimensionless quantity.

In order to simplify notation, a bar  $\bar{\cdot}$  will be dropped and dimensionless quantities:  $\bar{x}$ ,  $\bar{t}$ ,  $\bar{\sigma}$ ,  $\bar{u}$ ,  $\bar{\tau}$ ,  $\bar{u}_0$  and  $\bar{v}_0$  will be written as:  $x, t, \sigma, u, \tau, u_0$  and  $v_0$ . Obviously, condition  $0 < \tau_\sigma < \tau_\varepsilon$  implies  $0 < \tau < 1$ .

In our work an appropriate setting for studying system (1-3) will be distributional one. In fact, we will look for a fundamental solution to the generalized Cauchy problem for FZWE (4) in  $\mathcal{S}'(\mathbb{R} \times \mathbb{R}_+)$ . This suffices to obtain a solution to (1-3), since (1) and (4) are equivalent. Indeed, by applying the Laplace transform with respect to time variable  $t$  to the second equation in (1), one obtains

$$(1 + \tau s^\alpha) \tilde{\sigma}(x, s) = (1 + s^\alpha) \tilde{\varepsilon}(x, s).$$

According to [15],  $\mathcal{L}^{-1} \left( \frac{1+s^\alpha}{1+\tau s^\alpha} \right)$  is well-defined element in  $\mathcal{S}'_+(\mathbb{R})$ , hence

$$\sigma = \mathcal{L}^{-1} \left( \frac{1+s^\alpha}{1+\tau s^\alpha} \right) *_t \varepsilon. \quad (13)$$

Inserting  $\varepsilon$  from the third equation in (1) into (13) and then  $\sigma$  into the first equation in (1) yields

$$\frac{\partial^2}{\partial t^2} u(x, t) = \mathcal{L}^{-1} \left( \frac{1+s^\alpha}{1+\tau s^\alpha} \right) *_t \frac{\partial^2}{\partial x^2} u(x, t). \quad (14)$$

Setting  $L(t) = \mathcal{L}^{-1} \left( \frac{1+s^\alpha}{1+\tau s^\alpha} \right) *_t$  we come to (4). Therefore we have proved that (1) and (4) are equivalent. Notice that equation (14) is of the form  $Pu = 0$ , with

$$P := \frac{\partial^2}{\partial t^2} - \mathcal{L}^{-1} \left( \frac{1+s^\alpha}{1+\tau s^\alpha} \right) *_t \frac{\partial^2}{\partial x^2}. \quad (15)$$

## 4 The existence and uniqueness of a solution to the Cauchy problem (4), (2)

The aim of this section is to find a solution to the generalized Cauchy problem to (4), i.e.

$$\frac{\partial^2}{\partial t^2} u(x, t) = \mathcal{L}^{-1} \left( \frac{1+s^\alpha}{1+\tau s^\alpha} \right) *_t \frac{\partial^2}{\partial x^2} u(x, t) + u_0(x) \delta'(t) + v_0(x) \delta(t), \quad (16)$$

or equivalently

$$Pu(x, t) = u_0(x) \delta'(t) + v_0(x) \delta(t),$$

where  $P$  is given by (15).

**Remark 4.1.** We have already explained that initial conditions are included into the generalized Cauchy problem (see Section 2). For functions in  $\mathcal{S}'$  boundary conditions (3) are automatically fulfilled.

We state the main theorem.

**Theorem 4.2.** *Let  $u_0, v_0 \in \mathcal{S}'(\mathbb{R})$ . Then there exists a unique solution  $u \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}_+)$  to (16) given by*

$$u(x, t) = S(x, t) *_t (u_0(x) \delta'(t) + v_0(x) \delta(t)), \quad (17)$$

where

$$\begin{aligned} S(x, t) = & 1 + \frac{1}{4\pi i} \int_0^\infty \left( \sqrt{\frac{1+\tau q^\alpha e^{i\alpha\pi}}{1+q^\alpha e^{i\alpha\pi}}} e^{|x|q\sqrt{\frac{1+\tau q^\alpha e^{i\alpha\pi}}{1+q^\alpha e^{i\alpha\pi}}}} \right. \\ & \left. - \sqrt{\frac{1+\tau q^\alpha e^{-i\alpha\pi}}{1+q^\alpha e^{-i\alpha\pi}}} e^{|x|q\sqrt{\frac{1+\tau q^\alpha e^{-i\alpha\pi}}{1+q^\alpha e^{-i\alpha\pi}}}} \right) \frac{e^{-qt}}{q} dq, \end{aligned} \quad (18)$$

is the fundamental solution of operator  $P$ ,  $S \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}_+)$  with support in the cone  $|x| < \frac{t}{\sqrt{\tau}}$ .

We will need the following lemma.

**Lemma 4.3.** *Let  $f \in \mathcal{S}'(\mathbb{R})$ . Then the equation*

$$v'' - \omega v = -f \quad (19)$$

has a solution  $v \in \mathcal{S}'(\mathbb{R})$  for all  $\omega \in \mathbb{C} \setminus (-\infty, 0]$ , which is of the form

$$v = \frac{e^{-\sqrt{\omega}|x|}}{2\sqrt{\omega}} * f,$$

where  $\sqrt{\omega}$  is the main branch.

**Proof.** We first apply Fourier transform to (19):

$$\widehat{v} = \frac{1}{\xi^2 + \omega} \widehat{f}. \quad (20)$$

In order to find  $v$  we need to calculate

$$\mathcal{F}^{-1}\left(\frac{1}{\xi^2 + \omega}\right)(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} \frac{1}{\xi^2 + \omega} d\xi.$$

Let  $x \geq 0$ . Let  $\Gamma_+ = \Gamma_1 \cup \Gamma_2$ , where  $\Gamma_1 = \{z = Re^{i\varphi} \mid R > 0, 0 < \varphi < \pi\}$  and  $\Gamma_2 = \{z = \xi \mid -R < \xi < R\}$ . By the Cauchy residual theorem we have that

$$\oint_{\Gamma} e^{ixz} \frac{1}{z^2 + \omega} dz = 2\pi i \sum_{i=1}^k \text{Res}(z_i),$$

where  $z_i$  are poles of the function  $z \mapsto \frac{e^{ixz}}{z^2 + \omega}$  lying inside of  $\Gamma$ . For  $\omega \in \mathbb{C} \setminus (-\infty, 0]$  there is only one such pole  $z_1 = i\sqrt{\omega}$  and  $\text{Res}(z_1) = \frac{e^{ixz}}{2z} \Big|_{z=z_1} = \frac{e^{-x\sqrt{\omega}}}{2i\sqrt{\omega}}$ . Letting  $R \rightarrow \infty$ , the integral over  $\Gamma_1$  tends to zero and  $\int_{\Gamma_2} e^{ixz} \frac{1}{z^2 + \omega} dz = \int_{\mathbb{R}} e^{ix\xi} \frac{1}{\xi^2 + \omega} d\xi$ , thus

$$\mathcal{F}^{-1}\left(\frac{1}{\xi^2 + \omega}\right)(x) = \frac{1}{2\pi} \int_{\Gamma_2} e^{ixz} \frac{1}{z^2 + \omega} dz = \frac{e^{-x\sqrt{\omega}}}{2\sqrt{\omega}}.$$

For  $x < 0$  and  $\Gamma_- = \Gamma_3 \cup \Gamma_{eq:4}$ , where  $\Gamma_3 = \{z = Re^{i\varphi} \mid R > 0, -\pi < \varphi < 0\}$  and  $\Gamma_4 = \{z = -\xi \mid -R < \xi < R\}$  we can apply the same arguments as for the case  $x \geq 0$  to obtain  $\mathcal{F}^{-1}\left(\frac{1}{\xi^2 + \omega}\right)(x) = \frac{e^{x\sqrt{\omega}}}{2\sqrt{\omega}}$ .

Thus for  $x \in \mathbb{R}$  we can write

$$\mathcal{F}^{-1}\left(\frac{1}{\xi^2 + \omega}\right)(x) = \frac{e^{-\sqrt{\omega}|x|}}{2\sqrt{\omega}}.$$

Now, the inverse Fourier transform applied to (20) yields the result.  $\square$

**Proof of the main theorem.** Applying Laplace transform with respect to  $t$  to (16) we obtain:

$$\frac{\partial^2}{\partial x^2} \tilde{u}(x, s) - s^2 \frac{1 + \tau s^\alpha}{1 + s^\alpha} \tilde{u}(x, s) = -\frac{1 + \tau s^\alpha}{1 + s^\alpha} (su_0(x) + v_0(x)). \quad (21)$$

Set  $\omega(s) := s^2 \frac{1 + \tau s^\alpha}{1 + s^\alpha}$  and  $f(x, s) := \frac{1 + \tau s^\alpha}{1 + s^\alpha} (su_0(x) + v_0(x))$ ,  $s \in \mathbb{C}_+$ . In order to apply Lemma 4.3 to (21) we have to show first that  $f(\cdot, s) \in \mathcal{S}'(\mathbb{R})$ , which follows from assumptions that  $u_0, v_0 \in \mathcal{S}'(\mathbb{R})$  and second, that  $\omega(s) \in \mathbb{C} \setminus (-\infty, 0]$ , for all  $s \in \mathbb{C}_+$ . The latter can be verified in the following way. Take arbitrary  $s = \rho e^{i\varphi}$ ,  $\rho > 0$ ,  $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$ . Then  $\omega(s) = \rho^2 e^{2i\varphi} \frac{1 + \tau \rho^\alpha e^{i\alpha\varphi}}{1 + \rho^\alpha e^{i\alpha\varphi}}$ , and after a straightforward calculation we arrive to

$$\text{Re } \omega(s) = \frac{\rho^2}{A} [(1 + \rho^\alpha(1 + \tau) \cos(\alpha\varphi) + \tau \rho^{2\alpha}) \cos(2\varphi) + \rho^\alpha(1 - \tau) \sin(\alpha\varphi) \sin(2\varphi)], \quad (22)$$

$$\text{Im } \omega(s) = \frac{\rho^2}{A} [(1 + \rho^\alpha(1 + \tau) \cos(\alpha\varphi) + \tau \rho^{2\alpha}) \sin(2\varphi) - \rho^\alpha(1 - \tau) \sin(\alpha\varphi) \cos(2\varphi)], \quad (23)$$

where  $A = (1 + \rho^\alpha \cos(\alpha\varphi))^2 + \rho^{2\alpha} \sin^2(\alpha\varphi) > 0$ . Suppose that  $\omega(s) \in (-\infty, 0]$ , for some  $s \in \mathbb{C}_+$ . Then  $\text{Im } \omega(s) = 0$ , and (23) yields

$$(1 + \rho^\alpha(1 + \tau) \cos(\alpha\varphi) + \tau \rho^{2\alpha}) = \rho^\alpha(1 - \tau) \sin(\alpha\varphi) \frac{\cos(2\varphi)}{\sin(2\varphi)}. \quad (24)$$

From (22) we now obtain

$$\text{Re } \omega(s) = \frac{\rho^2}{A} \rho^\alpha(1 - \tau) \sin(\alpha\varphi) \sin(2\varphi) \left( \frac{\cos^2(2\varphi)}{\sin^2(2\varphi)} + 1 \right) \leq 0.$$

where we have used (24) and the assumption  $\omega(s) \in (-\infty, 0]$ , for all  $s \in \mathbb{C}_+$ .

Since  $\frac{\rho^2}{A} \rho^\alpha (1 - \tau) \sin(\alpha\varphi) \left( \frac{\cos^2(2\varphi)}{\sin^2(2\varphi)} + 1 \right) > 0$ , for  $\rho > 0$ ,  $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$ , it follows that  $\sin(2\varphi) < 0$ , and therefore  $\varphi \in (-\frac{\pi}{2}, -\frac{\pi}{4}] \cup [\frac{\pi}{4}, \frac{\pi}{2})$ . However, (24) can not be satisfied for those  $\varphi$ . (Indeed, for  $\varphi \in (\frac{\pi}{2}, -\frac{\pi}{4}]$ ,  $\cos(\alpha\varphi), \cos(2\varphi), \sin(\alpha\varphi) > 0$ , but  $\sin(2\varphi) < 0$  and similarly for  $\varphi \in [\frac{\pi}{4}, \frac{\pi}{2})$ .) Hence,  $\omega(s) \in \mathbb{C} \setminus (-\infty, 0]$ , for all  $s \in \mathbb{C}_+$ .

Now we can apply Lemma 4.3 to obtain a solution of (21):

$$\begin{aligned} \tilde{u}(x, s) &= \frac{e^{-\sqrt{\omega(s)}|x|}}{2\sqrt{\omega(s)}} *_x \frac{\omega(s)}{s^2} (su_0(x) + v_0(x)) \\ &= \frac{\sqrt{\omega(s)}e^{-\sqrt{\omega(s)}|x|}}{2s^2} *_x (su_0(x) + v_0(x)). \end{aligned} \quad (25)$$

To obtain a solution  $u$  to (4) we need to calculate inverse Laplace transform of (25). For that purpose set

$$\tilde{S}(x, s) := \frac{\sqrt{\omega(s)}e^{-\sqrt{\omega(s)}|x|}}{2s^2} = \frac{1}{2s} \sqrt{\frac{1 + \tau s^\alpha}{1 + s^\alpha}} e^{-|x|s\sqrt{\frac{1 + \tau s^\alpha}{1 + s^\alpha}}}, \quad x \in \mathbb{R}, s \in \mathbb{C}_+, \quad (26)$$

and

$$S(x, t) = \mathcal{L}^{-1} [\tilde{S}(x, s)](t), \quad x \in \mathbb{R}, t > 0.$$

$S(x, t)$  is well-defined since  $|\tilde{S}(x, s)| \leq C \frac{(1 + |s|^m)}{|\operatorname{Re} s|}$ ,  $m \in \mathbb{N}$ , so its inverse Laplace transform exists (cf. Section 2).

Note also that  $S$  is a fundamental solution of  $P$ . Then the solution  $u$  is given by (17).

Multiform function  $\tilde{S}$ , given by (26), has branch points at  $s = 0$  and  $s = \infty$  and has no singularities. Hence, it can be evaluated by the use of the Cauchy integral formula

$$\oint_{\Gamma} \tilde{S}(x, s) e^{st} ds = 0, \quad x \in \mathbb{R}, t > 0, \quad (27)$$

where  $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_\varepsilon \cup \Gamma_3 \cup \Gamma_4 \cup \gamma_0$ , is a contour given in Figure 1.

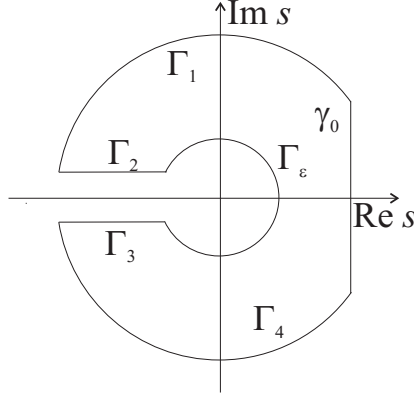


Figure 1: Integration contour  $\Gamma$

More precisely, for arbitrarily chosen  $R > 0$ ,  $0 < \varepsilon < R$  and  $a > 0$ ,  $\Gamma$  is defined by

$$\begin{aligned} \Gamma_1 : & s = Re^{i\varphi}, \varphi_0 < \varphi < \pi; \\ \Gamma_2 : & s = qe^{i\pi}, -R < -q < -\varepsilon; \\ \Gamma_\varepsilon : & s = \varepsilon e^{i\varphi}, -\pi < \varphi < \pi; \\ \Gamma_3 : & s = qe^{-i\pi}, \varepsilon < q < R; \\ \Gamma_4 : & s = Re^{i\varphi}, -\pi < \varphi < -\varphi_0; \\ \gamma_0 : & s = a(1 + i \tan \varphi), -\varphi_0 < \varphi < \varphi_0, \end{aligned}$$



where  $\varphi_0 = \arccos(\frac{a}{R})$ . Note that  $\lim_{R \rightarrow \infty} \varphi_0 = \frac{\pi}{2}$ . In the limit when  $R \rightarrow \infty$ , integral along contour  $\Gamma_1$  reads ( $x \in \mathbb{R}, t > 0$ )

$$\lim_{R \rightarrow \infty} \int_{\Gamma_1} \tilde{S}(x, s) e^{st} ds = \frac{1}{2} \lim_{R \rightarrow \infty} \int_{\varphi_0}^{\pi} \sqrt{\frac{1 + \tau R^\alpha e^{i\alpha\varphi}}{1 + R^\alpha e^{i\alpha\varphi}}} e^{-|x|Re^{i\varphi} \sqrt{\frac{1 + \tau R^\alpha e^{i\alpha\varphi}}{1 + R^\alpha e^{i\alpha\varphi}} + Rte^{i\varphi}}} i d\varphi. \quad (28)$$

Evaluating the absolute value of  $\int_{\Gamma_1} \tilde{S}(x, s) e^{st} ds$  one obtains ( $x \in \mathbb{R}, t > 0$ )

$$\lim_{R \rightarrow \infty} \left| \int_{\Gamma_1} \tilde{S}(x, s) e^{st} ds \right| \leq \frac{1}{2} \lim_{R \rightarrow \infty} \int_{\varphi_0}^{\pi} \left| \sqrt{\frac{1 + \tau R^\alpha e^{i\alpha\varphi}}{1 + R^\alpha e^{i\alpha\varphi}}} \right| e^{-|x|Re^{i\varphi} \sqrt{\frac{1 + \tau R^\alpha e^{i\alpha\varphi}}{1 + R^\alpha e^{i\alpha\varphi}}}} e^{Rt \cos \varphi} d\varphi.$$

In the limit when  $R \rightarrow \infty$  the expression  $\sqrt{\frac{1 + \tau R^\alpha e^{i\alpha\varphi}}{1 + R^\alpha e^{i\alpha\varphi}}}$  tends to  $\sqrt{\tau}$  and therefore

$$\lim_{R \rightarrow \infty} \left| \int_{\Gamma_1} \tilde{S}(x, s) e^{st} ds \right| \leq \frac{\sqrt{\tau}}{2} \lim_{R \rightarrow \infty} \int_{\varphi_0}^{\pi} e^{R \cos \varphi (t - |x|\sqrt{\tau})} d\varphi = 0, \quad \text{if } t > |x|\sqrt{\tau},$$

since  $\cos \varphi < 0$  for  $\varphi \in (\frac{\pi}{2}, \pi)$ . Similar argument is valid for the integral along  $\Gamma_4$ .

In the limit when  $\varepsilon \rightarrow 0$ , the integral along  $\Gamma_\varepsilon$  is given by the formula similar to (28) and it is calculated as

$$\lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon} \tilde{S}(x, s) e^{st} ds = \frac{1}{2} \lim_{\varepsilon \rightarrow 0} \int_{\pi}^{-\pi} \sqrt{\frac{1 + \tau \varepsilon^\alpha e^{i\alpha\varphi}}{1 + \varepsilon^\alpha e^{i\alpha\varphi}}} e^{-|x|\varepsilon e^{i\varphi} \sqrt{\frac{1 + \tau \varepsilon^\alpha e^{i\alpha\varphi}}{1 + \varepsilon^\alpha e^{i\alpha\varphi}} + \varepsilon t e^{i\varphi}}} i d\varphi = -i\pi.$$

Integrals along contours:  $\Gamma_2$ ,  $\Gamma_3$  and  $\gamma_0$ , in the limit when  $R \rightarrow \infty$ ,  $\varepsilon \rightarrow 0$ , give

$$\begin{aligned} \lim_{\substack{R \rightarrow \infty \\ \varepsilon \rightarrow 0}} \int_{\Gamma_2} \tilde{S}(x, s) e^{st} ds &= -\frac{1}{2} \int_0^\infty \sqrt{\frac{1 + \tau q^\alpha e^{i\alpha\pi}}{1 + q^\alpha e^{i\alpha\pi}}} e^{-q(t - |x|\sqrt{\frac{1 + \tau q^\alpha e^{i\alpha\pi}}{1 + q^\alpha e^{i\alpha\pi}}})} \frac{dq}{q} \\ \lim_{\substack{R \rightarrow \infty \\ \varepsilon \rightarrow 0}} \int_{\Gamma_3} \tilde{S}(x, s) e^{st} ds &= \frac{1}{2} \int_0^\infty \sqrt{\frac{1 + \tau q^\alpha e^{-i\alpha\pi}}{1 + q^\alpha e^{-i\alpha\pi}}} e^{-q(t - |x|\sqrt{\frac{1 + \tau q^\alpha e^{-i\alpha\pi}}{1 + q^\alpha e^{-i\alpha\pi}}})} \frac{dq}{q} \\ \lim_{R \rightarrow \infty} \int_{\gamma_0} \tilde{S}(x, s) e^{st} ds &= 2\pi i S(x, t). \end{aligned}$$

Now, by the Cauchy integral formula (27), we obtain  $S$  as in (18). Therefore we have proved the theorem.  $\square$

As a consequence of what we proved in Sections 3 and 4, we have the following corollary.

**Corollary 4.4.** *Let  $u$  be given by (17). Then*

$$(u, \varepsilon, \sigma)(x, t) = \left( u(x, t), \frac{\partial}{\partial x} u(x, t), \mathcal{L}^{-1} \left( \frac{1 + s^\alpha}{1 + \tau s^\alpha} \right) *_t \frac{\partial}{\partial x} u(x, t) \right) \in (\mathcal{S}'(\mathbb{R} \times \mathbb{R}_+))^3,$$

*is a unique solution to the system (1-3).*

In a similar way one can also consider nonhomogeneous case.

**Remark 4.5.** Let the first equation in (1) be replaced by

$$\frac{\partial}{\partial x} \sigma(x, t) = \frac{\partial^2}{\partial t^2} u(x, t) + f(x, t),$$

where  $f \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}_+)$ . This is a case of a rod under the influence of body forces. Then, the solution of the generalized Cauchy problem

$$\frac{\partial^2}{\partial t^2} u(x, t) = \mathcal{L}^{-1} \left( \frac{1 + s^\alpha}{1 + \tau s^\alpha} \right) *_t \frac{\partial^2}{\partial x^2} u(x, t) + f(x, t) + u_0(x) \delta'(t) + v_0(x) \delta(t),$$

is

$$u(x, t) = S(x, t) *_{x,t} (f(x, t) + u_0(x)\delta'(t) + v_0(x)\delta(t))$$

where  $S$  is as in (18).

**Remark 4.6.** Dimensionless condition  $|x| < \frac{t}{\sqrt{\tau}}$ , or  $|x| < \sqrt{\frac{\rho}{E}} \sqrt{\frac{\tau \alpha}{\tau_\epsilon}} t$  in the dimensional form, can be physically interpreted as the wave property. Namely, in the moment  $t$ , wave caused by the initial disturbance  $u_0(x) = \delta(x)$ ,  $x \in \mathbb{R}$ , has reached the point which is at distance  $|x|$  from the origin, where the initial disturbance was applied. Thus, constant  $c_Z = \sqrt{\frac{\rho}{E}} \sqrt{\frac{\tau \alpha}{\tau_\epsilon}}$  can be interpreted as the wave speed in the fractional viscoelastic media of Zener type.

## 5 A numerical example

Let  $u_0 = \delta$  and  $v_0 = 0$  in the solution (17) to the Cauchy problem of wave equation for fractional Zener type viscoelastic media. Then solution reads

$$u(x, t) = \frac{\partial}{\partial t} S(x, t), \quad x \in \mathbb{R}, t > 0, \quad (29)$$

i.e. the fundamental solution represents the solution itself. Figure 2 presents the plots of  $u(x, t)$ ,  $x \in (0, 3)$ ,  $t \in \{0.5, 1, 1.5\}$ , given by (29), for the set of parameters:  $\alpha = 0.23$  and  $\tau = 0.004$ .

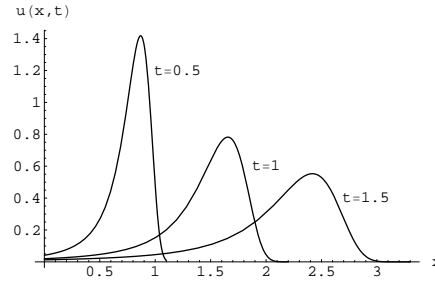


Figure 2: Solution  $u(x, t)$ ,  $x \in (0, 3)$ ,  $t \in \{0.5, 1, 1.5\}$

In order to see the effect of a change in the order of a fractional derivative, we present following figures. Figure 3 presents plots of  $u(x, t)$ ,  $x \in (0, 3)$ ,  $t \in \{0.5, 1, 1.5\}$ , given by (29), for different values of parameter  $\alpha$ . Plot of  $u$ , denoted by the dashed line corresponds to  $\alpha = 0.25$ , dot-dashed line is used for  $\alpha = 0.5$ , while the solid line denotes the plot for  $\alpha = 0.75$ . Parameter  $\tau = 0.004$  is the same in all three figures.

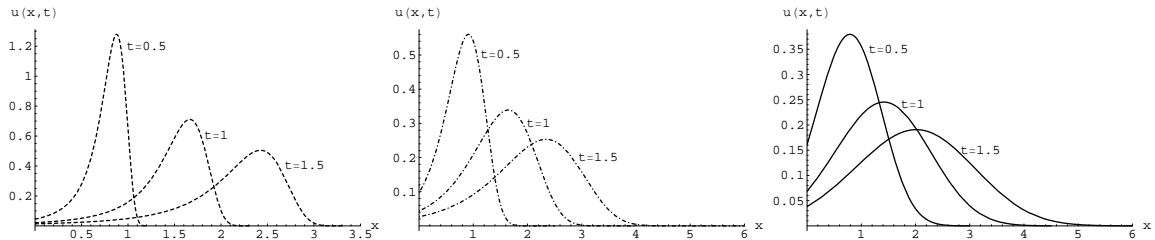


Figure 3: Solution  $u(x, t)$ ,  $x \in (0, 6)$ ,  $t \in \{0.5, 1, 1.5\}$ ,  $\alpha \in \{0.25, 0.5, 0.75\}$

From figure 3 one can see that for each value of the order of fractional derivative  $\alpha$ , as time increases, the maximum value of  $u$  decreases, which is the consequence of the dissipative model of a media.

Figure 4 presents plots of  $u(x, t)$ ,  $x \in (0, 3)$ ,  $\alpha \in \{0.25, 0.5, 0.75\}$ , given by (29), for different time moments, i.e. for  $t \in \{0.5, 1, 1.5\}$ . Parameter  $\tau$  and line styles are the same as in figure 3.

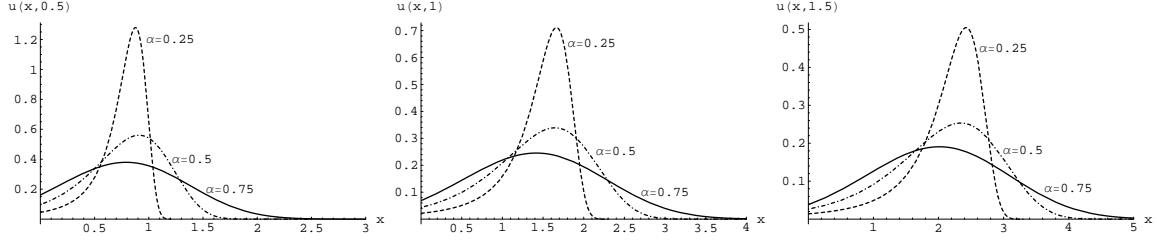


Figure 4: Solution  $u(x, t)$ ,  $x \in (0, 5)$ ,  $t \in \{0.5, 1, 1.5\}$ ,  $\alpha \in \{0.25, 0.5, 0.75\}$

From figure 4 one can see that at the same time instance, as the value of  $\alpha$  increases, the maximum value of  $u$  decreases, while the width of the maximum increases. This is the consequence of the fact that the constitutive equation of the fractional type in (1) describes media that tends to be elastic as  $\alpha$  tends to zero. Therefore, the fundamental solution (29) should tend to the Dirac distribution as  $\alpha \rightarrow 0$ . Also, media tends to be viscoelastic as  $\alpha$  in (1) tends to 1. Therefore, the fundamental solution (29) should tend to the solution of wave equation for viscoelastic media as  $\alpha \rightarrow 1$ .

Figure 5 presents plots of  $u$  for both different time moments and values of  $\alpha$ . Parameter  $\tau$  and line styles are the same as in figure 3.

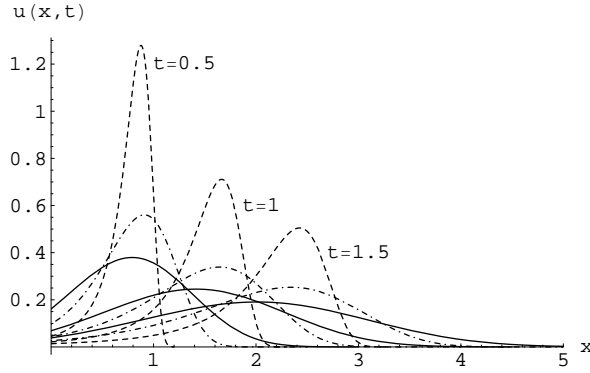


Figure 5: Solution  $u(x, t)$ ,  $x \in (0, 5)$ ,  $t \in \{0.5, 1, 1.5\}$ ,  $\alpha \in \{0.25, 0.5, 0.75\}$

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## References

- [1] T. M. Atanackovic, A modified Zener model of viscoelastic body, *Contin. Mech. Thermodyn.* 14 (2002) 137–148.
- [2] T. M. Atanackovic, A. Guran, *Theory of Elasticity for Scientists and Engineers*, Birkhäuser, Boston, 2000.

- [3] T. M. Atanackovi, S. Pilipovic, D. Zorica, Diffusion wave equation with two fractional derivatives of different order, *J. Phys. A: Math. Gen.* 40 (2007) 5319–5333.
- [4] T. M. Atanackovic, S. Pilipovic, D. Zorica, Time distributed-order diffusion-wave equation. I. Volterra-type equation, *Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci.* 465 (2009) 1869–1891.
- [5] T. M. Atanackovic, S. Pilipovic, D. Zorica, Time distributed-order diffusion-wave equation. II. Applications of Laplace and Fourier transformations, *Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci.* 465 (2009) 1893–1917.
- [6] M. Caputo, F. Mainardi, A new dissipation model based on memory mechanism, *Pure Appl. Geophysics* 91 (1971) 134–147.
- [7] M. Caputo, F. Mainardi, Linear models of dissipation in anelastic solids, *Riv. Nuovo Cimento (Ser. II)* 1 (1971) 161–198.
- [8] R. Dautry, J. L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*, volume 5, Springer-Verlag, Berlin, 2000.
- [9] V. L. Gonsovski, Yu. A. Rossikhin, Stress waves in a viscoelastic medium with a singular hereditary kernel, *J. Appl. Mech. Tech. Phys.* 14 (1973) 595–597.
- [10] A. Hanyga, Multi-dimensional solutions of space-time-fractional diffusion equations, *Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci.* 458 (2002) 429–450.
- [11] A. Hanyga, Multidimensional solutions of time-fractional diffusion-wave equations, *Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci.* 458 (2002) 933–957.
- [12] A. N. Kochubei, Distributed order calculus and equations of ultraslow diffusion, *J. Math. Anal. Appl.* 340 (2008) 252–281.
- [13] F. Mainardi, Fractional calculus: Some basic problems in continuum and statistical mechanics, In F. Mainardi, A. Carpinteri, editors, *Fractals and Fractional Calculus in Continuum Mechanics*, volume 378 of *CISM Courses and Lectures*, pages 291–348, Springer-Verlag, Wien and New York, 1997.
- [14] F. Mainardi, G. Pagnini, R. Gorenflo, Some aspects of fractional diffusion equations of single and distributed order, *Appl. Math. Comput.* 187 (2007) 295–305.
- [15] Lj. Oparnica, Generalized fractional calculus with applications in mechanics. *Matematički vesnik* 53 (2002) 151–158.
- [16] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
- [17] Yu. N. Rabotnov, Equilibrium of an elastic medium with after effect, (in Russian), *Prikl. Matem. i Mekh.* 12 (1948) 81–91.
- [18] Yu. A. Rossikhin, M. V. Shitikova, A new method for solving dynamic problems of fractional derivative viscoelasticity, *Int. J. Eng. Sci.* 39 (2001) 149–176.
- [19] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives - Theory and Applications*, Gordon and Breach Science Publishers, Amsterdam, 1993.
- [20] F. Trèves, *Basic Linear Partial Differential Equations*, Academic press, New York, 1975.
- [21] V. S. Vladimirov, *Equations of Mathematical Physics*, Mir Publishers, Moscow, 1984.